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Limits of Convolution Iterates and Properties of Random Walks Defined on Regular Semigroups with Applications to Matrix Semigroups

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We consider the properties of a random walk $Z_n = X_1 X_2 \cdots$ defined on a topological semigroup S which has the additional property that every element of S has a generalized inverse. We give conditions for when Z_n is recurrent and also consider the value of $\alpha = \sup_{K \subset S,\ K \text{ compact}} \lim_{n \to \infty} \sup_{x \in S} \mu^n(Kx^{-1})$. Since every matrix has at least one generalized inverse, the techniques presented will be applied to semigroups of matrices. © 1988 Academic Press, Inc.

1. INTRODUCTION

Products of random matrices have recently been applied to a wide variety of real world situations (see Girko [4], Titubalin [14, 15], and Molchanov and Titubalin [6]). Usually, the random matrix product is reduced to a product on the real line (see Pincus [11] and Girko [4]). Several papers, however, have been written considering semigroups of $n \times n$ matrices (see Nakassis [9], Sun [13], and Mukherjea and Nakassis [7]). In particular, Hognas and Mukherjea [5] have written an extremely profound paper concerning the recurrence of random walks on semigroups of matrices.

In this paper, we consider recurrence properties and limits of convolution iterates in the more general context of first completely regular and then regular semigroups. Then we apply the results to the specific case of matrix semigroups.

Before beginning, it is necessary to state some basic algebraic properties concerning regular semigroups and also some basic probabilistic properties concerning topological semigroups. For more information for the first, see Clifford and Preston [2] and Petrich [10]. For more information concerning measures see Mukherjea and Tserpes [8].

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An element $x \in S$ is regular provided that $x \in xSx$. That is, there exists some $y \in S$ such that $x = xyx$. A semigroup is regular if every element $x \in S$ is regular. If S is regular, then it has the following properties:

1) If $x \in S$ and $xyx = x$ for some $y \in S$ then $e = xy$ and $f = yx$ are idempotent elements of S.

2) The principal right (left) ideal generated by $x \in S$ is equal to $xS(Sx)$.

3) For $x \in S$ there exists an idempotent element $e = xy$ such that $xS = eS(Sx = Sf$ for $f = yx)$.

4) If $s \in S$ then a generalized inverse of x is any element z such that $xzx = x$ and $zxz = z$. Every element in S has at least one inverse; if $xyx = x$ then $z = yxy$ is an inverse of x.

5) If e, f, ef, and fe are idempotent elements of S then ef and fe are inverses of each other.

6) If A is a right ideal of S and B is a left ideal then $A \cap B = AB$.

7) A regular semigroup containing exactly one idempotent is a group.

8) The element $x \in S$ is completely regular if there exists $y \in S$ such that $x = xyx$ and $xy = yx$. S is completely regular if all elements of S are completely regular.

9) S is a union of maximal pairwise disjoint subgroups if and only if S is completely regular.

Let S be a locally compact, Hausdorff, second countable topological semigroup. Then a measurable function from a probability space into S

$$X: (\Omega, \xi, P) \to S$$

is a random variable defined in S. If for any measurable subset A of S,

$$\mu(A) = P\{\omega: X(\omega) \in A\}$$

then X has law μ. For any $x \in S$ and $A \subset S$, define

$$x^{-1}A = \{y \in S: xy \in A\}.$$

A right (left) random walk is equal to $Z_n = X_1 X_2 \cdots X_n$ $(L_n = X_n X_{n-1} \ldots X_1)$ where the $\{X_i\}$ are independent and identically distributed and $Z_n(L_n)$ has law μ^n where μ^n denotes the n-fold convolution of μ. For $x, y \in S$, x leads to $y(x \to y)$ with respect to Z_n provided that for any neighborhood N_y of y there exists some $n > 0$ such that

$$P(Z_n \in x^{-1}N_y) = P_x(Z_n \in N_y) = P(xZ_n \in N_y) = \mu^n(x^{-1}N_y) > 0.$$

If for all neighborhoods N_y of y

$$P_x(Z_n \in N_y \text{ i.o.}) = P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{xZ_n \in N_y\}\right) = 1$$

then $x \to y$ i.o. Otherwise if $x \to y$ but $x \not\to y$ i.o. then $x \to y$ f.o. If $x \to x$ i.o. then x is recurrent and $x \in S$ is essential provided $x \to y$ always implies that $y \to x$.

Throughout this paper we assume that the support S_μ of μ always generates S.

$$S = \overline{\bigcup_{n=1}^{\infty} S_\mu^n}$$

Then $x \to y$ if and only if $y \in \overline{xS}$.

2. Completely Regular Semigroups

Since we are assuming S is completely regular, for any $x \in S$ there exists $y \in S$ such that $x = xyx$ and $xy = yx$. Let $z = yxy$. Then z is an inverse element of x and

$$xz = x(yxy) = xy(xy) = yx(yx) = (yxy)\,x = zx$$

so that x commutes with at least one generalized inverse element. Also xz is an idempotent element in S.

We want to consider the properties of a right random walk Z_n defined in S. Although $L_n \neq Z_n$, its properties can be explored in the same fashion so it is only necessary to concentrate on Z_n. For random walks defined on the real line, the terms "essential" and "recurrent" are interchangeable. Unfortunately this is not true in a general semigroup. Therefore we begin by considering the properties of essential elements in S. To do so, we require the following lemmas which can easily be shown to hold for any regular semigroup:

Lemma 1. *For any $x \in S$, xS is closed with respect to the topology on S.*

Lemma 2. *For a right random walk Z_n, the following are equivalent for $x, y \in S$:*

1) $y \in xS$
2) $x \to y$
3) *For any idempotent e such that $x = ex$, $e \to y$*

4) *For any idempotent f such that $y = fy$, $x \to f$*

5) *If $x = ex$, $y = fy$ then $e \to f$*

Now we can show the following result:

THEOREM 1. *Let $e \in S$ be an essential idempotent and let $S_e = \{x \in S\colon e \to x\}$. Then $S_e = eS$ and eS is a completely regular semigroup consisting entirely of essential elements. Furthermore if $eS = Se$ then e is essential for both the left and right random walk. Moreoever eS is equal to the kernel of S, the maximal two sided ideal contained in S.*

Proof. If $x \in eS$ then by Lemma 2, $e \to x$ so that $eS \subset S_e$. Also by Lemma 2 if $e \to x$ then $x \in eS$ and $eS = S_e$. Suppose f is an idempotent element of S such that $f \in eS$. Then $fS \in eS$. But $e \to f$ implies $f \to e$ and $e \in fS$. This implies $fS = eS$. Let $x \in eS$ and assume $x \to y$. Then $y \in xS = fS$ for some idempotent $f \in S$. Since $xS = fS \subset eS$, $y \in eS$ and $e \to y$. Since e is essential, $y \to e$ and $y \in eS = xS$. Therefore $y \to x$ and x is essential.

To show that eS is completely regular we use the result from Petrich [10] that for any idempotent $g \in S$,

$$gS \cap Sg = G_g$$

is a maximal subgroup of S such that if f is any other idempotent in S,

$$G_g \cap G_f = 0$$

and

$$S = \bigcup_{\substack{g \in S \\ g \text{ idempotent}}} G_g.$$

Also, if S is any union of pairwise disjoint groups then S is completely regular. Let $f \in S_e$. Then $fS = eS$ so that $G_f \subset eS$. Also if $x \in S_e$ then $x \in G_f$ for some $f \in S_e$ and it is clear that

$$Se = \bigcup_{f \in Se} G_f$$

and Se is completely regular.

Suppose $Se = eS$. Then it is clear that if e is essential for the right random walk, it is also essential for the left random walk. Since $eS = eS \cap Se$, it is also clear that eS is a group. By Ellis' Theorem [8], it is a topological group. Also since

$$eSS = eS^2 \subset eS \qquad \text{and} \qquad SeS = (Se)\,S = (eS)\,S \subset eS$$

eS is a two sided ideal of S. Since no group contains a proper ideal, eS is the kernel of S. Q.E.D.

We will assume that S contains at least one essential element. First we will consider the case where there exists $e \in S$ such that e is essential and $eS = Se$. Then we will consider the situation where no such element exists.

When e is essential and $eS = Se$, eS is both a group and an ideal. Therefore, for any $n > 0$ and any $x \in eS$,

$$P(xZ_n \in eS) = 1.$$

Therefore we can show the following:

THEOREM 2. *Suppose $e \in S$ is essential such that $eS = Se$. Then if $\mu(eS) > 0$,*

$$\lim_{n \to \infty} \mu^n(eS) = 1$$

and no element of $(eS)^c$ can be recurrent.

Proof. Since eS is closed with respect to the topology defined on S, $(eS)^c$ is an open neighborhood of any element $x \in (eS)^c$. By the Borel-Cantelli lemma, if

$$\sum_{n=1}^{\infty} P_x(Z_n \in (eS)^c) < \infty$$

then

$$P_x(Z_n \in (eS)^c \text{ i.o.}) = 0$$

and x is not recurrent. Consider the set

$$x^{-1}(eS)^c = \{y: xy \notin eS\}.$$

If $y \in eS$ then $xy \in eS$ since eS is an ideal. Therefore $x^{-1}(eS)^c \subset (eS)^c$ and

$$\begin{aligned} P_x(Z_n \in (eS)^c) &= P(Z_n \in x^{-1}(eS)^c) \\ &\leqslant P(Z_n \in (eS)^c) \\ &= P(X_i \in (eS)^c \text{ for } i = 1, ..., n) \\ &= \prod_{i=1}^{n} P(X_i \in (eS)^c) \\ &= \mu((eS)^c)^n. \end{aligned}$$

Since $\mu((eS)^c) < 1$,

$$\sum_{n=1}^{\infty} P_x(Z_n \in (eS)^c) \leqslant \sum_{n=1}^{\infty} \mu((eS)^c)^n < \infty$$

and x is not recurrent. From this it is also clear that

$$\lim_{n \to \infty} \mu^n(eS) = 1. \qquad \text{Q.E.D.}$$

Using the fact that eS is an ideal, the following lemma can easily be shown:

LEMMA 3. *An element $x \in eS$ is recurrent if and only if for any neighborhood V defined entirely in eS,*

$$P_x(Z_n \in V \text{ i.o.}) = 1.$$

Using Lemma 3, the next three results can be shown by reducing the problem to the group eS. Since the proofs parallel the same arguments used for groups, they will be omitted.

THEOREM 3. *Suppose $\mu(eS) > 0$. Then for*

$$\alpha = \sup_{\substack{K \text{ compact} \\ K \subset S}} \limsup_{n \to \infty} \sup_{x \in S} \mu^n(Kx^{-1})$$

either $\alpha = 0$ or $\alpha = 1$. Moreover if $\alpha = 1$ there exists a sequence $\{a_n\}$ of elements in S such that for any $k = 0, 1, 2, \ldots$,

$$\mu^{n-k} * \delta_{a_n}$$

converges vaguely to a probability measure as $n \to \infty$ where δ_{a_n} denotes point mass.

Proof. See Csiszar [3].

THEOREM 4. *Suppose μ is a probability measure defined in S with $\mu(eS) > 0$. Suppose that there exists an open set V in eS with compact closure such that for every $x \in eS$; $x^{-1}Vx = V$. Then for every compact $K \subset S$,*

$$\sup\{\mu^n(Kx^{-1}) : x \in S\} \to 0 \qquad \text{as} \quad n \to \infty.$$

Proof. See Center and Mukherjea [1].

Before stating the next theorem we require a definition. Given an abelian topological group G, the dual group Γ of G consists of characters of G such that $\gamma \in \Gamma$ if

1) γ is a complex-valued function,

2) $|\gamma(x)| = 1$ for all $x \in G$, and

3) $\gamma(xy) = \gamma(x)\gamma(y)$ for $x, y \in G$.

THEOREM 5. *Assume that eS is abelian with $\mu(eS) > 0$. Then the random walk Z_n is transient if and only if there exists a neighborhood N of e defined in eS such that*

$$\overline{\lim_{t \uparrow 1}} \int_N \mathrm{Re}\left(\frac{1}{1 - t\phi(\gamma)}\right) d\gamma < \infty$$

where $\phi(\gamma) = \int_{eS} \gamma(g)\, d\mu(g)$.

Proof. See Revuz [12].

Remark 1. Although we required in the preceeding theorems that $\mu(eS) > 0$ this requirement can be removed. Since the support of μ generates S, there exists a minimal m for which $\mu^m(eS) > 0$. Then the argument in Theorem 2 may be replaced by

$$\begin{aligned} P_x(Z_n \in (eS)^c) &\leqslant \left[\prod_{i=1}^{k} P(X_1 \cdots X_m \in (eS)^c\right] P(X_1 \cdots X_j \in (eS)^c) \\ &= [\mu^m(eS)^c]^k \, \mu^j((eS)^c) \\ &= \mu^m((eS)^c)^k \cdot 1 \end{aligned}$$

where $n = mk + j$. Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} P_x(Z_n \in (eS)^c) \\ &\qquad = \sum_{k=0}^{\infty} \mu^m((eS)^c)^k \sum_{j=1}^{m-1} 1 \\ &\qquad = (m-1) \sum_{k=0}^{\infty} \mu^m((eS)^c)^k < \infty. \end{aligned}$$

We assume $\mu(eS) > 0$ merely for convenience.

Remark 2. Any abelian regular semigroup is completely regular. Therefore there can exist at most one essential idempotent e. If it exists then

$$\lim_{n \to \infty} \mu^n(eS) = 1$$

and the preceeding three theorems have characterized its properties. If e does not exist then no element of S can be recurrent.

Remark 3. Note that the recurrence properties depend completely on the algebraic properties of S. The element x is essential if and only if $y \in xS$ implies $x \in xS$. If e is essential and $eS = Se$ then no element outside eS can possibly be recurrent. Finally if $eS = Se$ then eS is a group and all the theory available for groups will work for eS.

We now turn our attention to completely regular semigroups where no such set exists. We begin with an example. Suppose

$$S = \{0, 1\}$$

$$0 \cdot 0 = 0 \cdot 1 = 0$$

$$1 \cdot 1 = 1 \cdot 0 = 1.$$

Then

$$S_0 = \{0, 1\} \qquad 0S = \{0\}$$

$$S_1 = \{0, 1\} \qquad 1S = \{1\}$$

$$G_0 = \{0\} \qquad G_1 = \{1\}$$

so it is clear that S is completely regular. Also 0 and 1 are both essential in S and $P_0(Z_n = 0) = P_1(Z_n = 1) = 1$ so that 0 and 1 are both recurrent for any measure on S. Also $\{0\}$ and $\{1\}$ are two disjoint essential classes in S. For the left random walk, $P_0(L_n = 0) = P(X_n \cdots X_1 0 = 0) = P(X_n = 0) = \mu(0)$ and

$$\begin{aligned} P_0(L_n = 0 \text{ i.o.}) &= \lim_{n \to \infty} P_0\left(\bigcap_{k=n}^{\infty} \{L_n = 0\}\right) \\ &= \lim_{n \to \infty} P_1(X_k = 0 \text{ for } k = n, n+1, \ldots) \\ &= \lim_{n \to \infty} \prod_{k=n}^{\infty} \mu(0) = 0. \end{aligned}$$

Similarly $P_1(L_n = 1) = 0$ so that $\{0, 1\}$ is an essential class that is not recurrent. Therefore for any measure μ, Z_n is recurrent but L_n is not. Note that if $eS = Se$ then L_n and Z_n have identical properties. However, using a similar argument we can also show

THEOREM 6. *Suppose S is completely regular. Let $e \in S$ be essential such that $eS \subset Se$. Then*

$$\lim_{n \to \infty} \mu^n(Se) = 1$$

and no element of $(Se)^c$ can be recurrent.

Suppose $x \in Se$. We want to find conditions for when $x \to y$ i.o. It is clear that y must be an element of Se. Consider $xZ_n = xX_1X_2 \cdots X_n$. Since $x \in Se$, $xZ_n \in Se$ for any n. We know that for any $z \in Se$, $ze = z$. Therefore

$$\begin{aligned} xX_1X_2 \cdots X_n &= (xe)(X_1e)(X_2e) \cdots (X_{n-1}e)\, X_n \\ &= x(X_1eX_2) \cdots (eX_{n-1})(eX_n). \end{aligned}$$

Thus

$$P_x(Z_n \in N_y \text{ i.o.}) = P(Z_n \in x^{-1}N_y \text{ i.o.}) = 0$$

if there exists some neighborhood N_y of y such that $x^{-1}N_y \cap eS = 0$ since Z_n is defined entirely in the group eS. Clearly if $x \in eS$, $\mu^n(x^{-1}(eS)) = 1$ using properties of groups, we have the following result:

THEOREM 7. *Suppose there exists some $x \in eS$ which is recurrent. Then every element of eS is recurrent. Moreover if for any $y, z \in Se$, $y^{-1}N_z \cap eS \neq 0$ for every neighborhood N_z of z then $y \to z$ i.o.*

From our example $S = \{0, 1\}$ it is clear that the above condition is not always satisfied. What is known however is that the sets G_g for g idempotent in S are also ideals of S. Therefore for $x \in G_g = SgS$, $P_x(Z_n \in G_g) = 1$ for every n. Therefore we can define a corresponding random walk Z_n' defined entirely in G_g and x is recurrent with respect to Z_n' if and only if it is recurrent with respect to Z_n. Also if the elements of G_g are recurrent then for any $x \in S$, if $x^{-1}N_x \cap G_g \neq 0$ for any neighborhood N_x of x then every element in G_f is recurrent where $x \in G_f$. Theorem 7 can be modified accordingly.

We close this section with a discussion of a completely regular semigroup where no essential elements exist. We consider a partial order defined on the idempotent elements of S,

$$f < e \text{ provided } f \in eS \text{ but } e \notin fS.$$

It is clear that $e \in S$ cannot be essential if there exists $f \in S$ such that $f < e$. However, if e is minimal with respect to this partial order then it must be essential.

THEOREM 8. *Assume S is completely regular and S has no essential element. Then no element of S is recurrent.*

Proof. Suppose $e \in S$ is idempotent and recurrent. Since e is not essential there exists $f \in S$ with $f < e$. Since $e \notin fS$, $(fS)^c$ is an open neighborhood of e. Suppose there exists $k > 0$ such that $\mu^k(fS) > 0$. Then

$$\begin{aligned} P_e(Z_n \in (fS)^c \text{ f.o.}) &\geqslant P_e(Z_n \in fS \text{ for all } n \geqslant k) \\ &= P_e(Z_k \in fS) \\ &= \mu^k(e^{-1}(fS)). \end{aligned}$$

Since $f \in eS$, $ef = f$ which makes it clear that $fS \subset e^{-1}(fS)$. Therefore

$$P_e(Z_n \in (fS)^c \text{ f.o.}) \geqslant \mu^k(fS) > 0$$

which contradicts the assumption that e is recurrent so $\mu^k(fS) = 0$ for all $k > 0$. But then $(fS)^c$ contains the support of μ and fS is not generated by this support. This again contradicts our assumption and e is not recurrent.

Now assume that x is any arbitrary element of S. Then for some $e \in S$, $xe = ex = x$ ($x \in G_e$ for some e). Therefore if $x \to x$ i.o., since $x \to e$,

$$\begin{aligned} 0 = P_x(Z_n \in N_x \text{ f.o.}) &\geqslant P_x(Z_k \in N_e, Z_n \in N_x \text{ f.o.}) \\ &= P_x(Z_k \in N_e, Z_{n-k} \in N_e^{-1} N_x \text{ f.o.}) \\ &\geqslant P_x(Z_k \in N_e, Z_{n-k} \in e^{-1} N_x \text{ f.o.}) \\ &= P_x(Z_k \in N_e)\, P_e(Z_{n-k} \in N_x \text{ f.o.}). \end{aligned}$$

There exists some k such that $P_x(Z_k \in N_e) > 0$. Therefore $P_e(Z_{n-k} \in N_x \text{ f.o.}) = 0$ and $e \to x$ i.o. But then

$$\begin{aligned} 0 = P_e(Z_n \in N_x \text{ f.o.}) &\geqslant P_e(Z_n \in N_x, Z_k \in N_e \text{ f.o.}) \\ &= P_e(Z_{n-k} \in N_e^{-1} N_x, Z_k \in N_e \text{ f.o.}) \\ &= P_e(Z_k \in N_e \text{ f.o.})\, P(Z_{n-k} \in N_e^{-1} N_x). \end{aligned}$$

Since $P(Z_{n-k} \in N_e^{-1} N_x) > 0$, where $n - k$ is constant $P_e(Z_k \in N_e \text{ f.o.}) = 0$ for any neighborhood N_e of e. However since e is idempotent this contradicts the first part of the argument that e is not recurrent. Therefore x is not recurrent. Q.E.D.

We can now put together the results of Theorem 7 and Theorem 8. For any idempotent $e \in S$, if e is not minimal with respect to the partial ordering on $E_S = \{\text{idempotent of } S\}$ then no element of G_e can be recurrent. However if e is minimal then $P_e(Z_n \in G_e) = 1$ so that the recurrence of elements in G_e can be determined entirely using group properties–Theorems 3, 4, and 5 can easily be shown for G_e. Thus the properties of Z_n depend entirely on its behavior in G_e where e is a minimal idempotent.

3. Regular Semigroups

The arguments in the preceding section relied very heavily on the group structure and also the ideal structure of a completely regular semigroup, neither of which are available in an arbitrary regular semigroup. The exception is Theorem 8, which used the partial ordering on the idempotents and the fact that $xS = eS$ for any $x \in S$. Therefore in any regular semigroup, if x is not essential then x cannot be recurrent.

To consider the recurrence properties of S we need to investigate the algebraic structure of eS where e is minimal with respect to the partial order on the idempotents. From property (6) in Section 1, $eS \cap Se$ is an ideal for any idempotent. Therefore for any $x \in eS \cap Se$,

$$P_x(X_1 \cdots X_n \in eS \cap Se) = 1$$

and the random walk Z_n induces a random walk Z'_n defined entirely in $eS \cap Se$. If $eS \cap Se$ is a group then its recurrence properties are identical with those of a group.

From a result in Clifford and Preston [2], it is known that so long as S does not have a zero element, since eS is a minimal right ideal for the essential idempotent e, $K = eS \cap Se$ is completely simple. As in the case when K is a group, Z_n induces a random walk Z'_n defined entirely in K. The properties of Z'_n have been completely studied in Mukherjea and Tserpes [8]. Therefore we have the following result:

THEOREM 9. *Assume that S does not contain a zero element. Suppose $e \in S$ is minimal with respect to the partial ordering defined on S. Then if S has no zero element, $eS \cap Se = SeS$ is the kernel K of S. Moreover, K is completely simple. The element $x \in K$ is recurrent with respect to the random walk $Z_n = X_1 X_2 \cdots X_n$ if and only if x is recurrent with respect to $Z'_n = Y_1 Y_2 \cdots Y_n$ where $Y_i = eX_i e \in K$ for all i.*

If S contains a zero element then $P_0(Z_n = 0) = 1$ for all n and any random walk Z_n defined on S. Therefore we wish to exclude it from consideration. Also, $0 < e$ for any idempotent $e \in S$ so we modify our partial ordering to add

$$e \text{ is 0-minimal in } S \text{ if } f < e \text{ implies } f = 0.$$

If $e \in S$ is 0-minimal then from Clifford and Preston [2], $eS \cap Se$ is completely 0-simple. If $x \in S$ such that $\mu^k(x^{-1}0) > 0$ for some k then for N_x such that $0 \notin N_x$,

$$P_x(Z_n \in N_x \text{ f.o.}) \geqslant P_x(Z_k = 0) > 0$$

and x cannot be recurrent. Therefore we only consider the set $K = eS \cap Se$ where e is 0-minimal with respect to our partial ordering and there exists $x \in K$ such that $\mu^k(x^{-1}0) = 0 = \mu^k(0x^{-1})$ for all $k > 0$ and $x^2 \neq 0$. Then the random walk Z_n defined in S induces a random walk Z'_n defined in K. By the following Theorem, we can define $K' \subset K$ such that $\mu^k(K/K') = 0$ for all $k > 0$ and we can induce a random walk Z''_n on the completely simple semigroup K'.

THEOREM 10. *Let e be 0-minimal in S. Let A_e = the nonzero idempotents in Se, B_e = the nonzero idempotents in eS, and $G_e = eSe - \{0\}$. Then if G_e is closed with respect to multiplication, it is a topological group. If there exists $x \in K$ such that $\mu^k(x^{-1}0) = \mu^k(0x^{-1}) = 0$ for all $k > 0$ and $x^2 \neq 0$ then $\mu^k(K - A_e B_e G_e) = 0$ for all $k > 0$. Therefore Z'_n is recurrent on K only if Z''_n is recurrent on $K' = A_e G_e B_e$.*

Proof. See Hognas and Mukherjea [5].

4. MATRIX SEMIGROUPS

The recurrence properties of random walks defined on semigroups of matrices were shown in Hognas and Mukherjea [5]. However, the arguments used required the concept of the rank of a matrix and cannot be generalized to other semigroups. On the other hand, the assumption that the support of μ generates S was not used. Instead this was simply overcome by using the Rees representation to induce a semigroup S' where the support of μ generates S'.

The results of Section 3 duplicate those of Hognas and Mukherjea [5] using the algebraic properties of regular semigroups, which were needed to show the existence of a completely (0-) simple kernel for S. In [5], the properties of the matrices themselves were used to show this existence. What we can also show is that for any semigroup S with a completely (0-) simple kernel, the recurrence of a random walk defined on S is completely determined by the induced random walk defined on K. These properties are categorized in Mukherjea and Tserpes [8] for the completely simple case. For the completely 0-simple case, the properties can easily be shown using the identical arguments of Hognas and Mukherjea [5].

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